Solving Combinatorial Optimization Problems

Combinatorial optimization problem

- A combinatorial optimization problem is a tuple (V, f, c)
- V is a set of *discrete* variables with *finite* domains
- An *assignment* maps each *v* Î *V* to a value in v's domain
- f is a function that decides *feasibility* of assignments
  - $f(a)$ returns *true* if and only if assignment a is feasible
- c is a function that returns the *cost* of an assignment
  - $c(a)$ is the cost of assignment a
  - assignment a_1 is *preferred* over assignment a_2 if $c\ (a_1) < c(a_2)$
- Problem:

  min $c(V)$ st $f(V)$

MI/MR as combinatorial optimization

- MI
  - variables: components with domains the possible modes
    - an assignment corresponds to a candidate diagnosis
  - feasibility: consistency with observations
  - cost: probability of a candidate diagnosis
- MR
  - variables: components with domains the possible modes
    - an assignment corresponds to a candidate repair
  - feasibility: entailment of goal
  - cost: cost of repair

Simple cost model

- Each variable has an associated cost of assigning it a value
  - $c(v_i = l_i)$ is the cost of assigning value l_i to variable v_i
- Cost of a complete assignment is the *sum* of the costs of the individual variable assignments
  - if assignment a is $v_1=l_1,\ldots,v_n=l_n$ then $c(a) = \Sigma_i\, c(v_i=l_i)$
- Costs of all variable values are non-negative
  - $c(v_i = l_i)$ • 0
- Each variable has a minimum cost value with cost 0
- Generating a least cost assignment is straightforward
  - each variable is assigned a value with cost 0

Using the simple cost model for MI

- Most probable diagnosis with *independent* component failures
  [de Kleer & Williams 89; de Kleer 91; Williams & Nayak 96]
  - $p(v_1=l_1,\ldots,v_n=l_n) = p(v_1=l_1)$ ´ … ´ $p(v_n=l_n)$
  - let m_i be the most probable mode for component v_i
  - $c(v_i=l_i) = -\log(p(v_i=l_i) / p(v_i=m_i))$
    - $\Rightarrow$all costs are non-negative with $c(v_i=m_i) = 0$
    - $\Rightarrow$for any assignments a_1 and a_2, $c(a_1)$ Š $c(a_2)$ iff $p(a_1)$ • $p(a_2)$
- Infinitesimal probabilities of *independent* failures
  [de Kleer 93; Pearl 92]
  - $k(v_i=l_i) = n$ means that $p(v_i=l_i)$ is $O(e^n)$ for infinitesimal $\boldsymbol{e}$
  - $k(v_1=l_1,\ldots, v_n=l_n) = k(v_1=l_1)+\ldots+k(v_n=l_n)$
  - $\Rightarrow$ let $c(v_i=l_i) = k(v_i=l_i)$
    - note: for each v_i there is an m_i such that $k(v_i=m_i) = 0$

Limitations of the simple cost model

- *Dependent* faults [Srinivas & Nayak 96]
  - probabilistic dependence between component failures captured using a Bayesian network
  - need to use a special enumeration algorithm

Best first search

- Used in [de Kleer & Williams 89; Dressler & Struss 94; Williams & Nayak 96]

```
function BFS(V, f, c)
    Initialize Agenda to a least cost assignment
    Initialize Solutions to the empty set
    while Agenda is non-empty do
        Let A be one of the least cost assignments in Agenda
        Remove A from Agenda
        if  f(A) is true then Add A to Solutions endif
        Add immediate successor assignments of A to Agenda
        if  enough solutions then return  Solutions endif
    endwhile
    return Solutions
  end BFS
```

Required subroutines for *BFS*

- Generating a least cost assignment
- Generating the immediate successors of an assignment
  - *completeness*: every feasible assignment must be the (eventual) successor of the least cost assignment
  - *monotonicity*: if *b* is an immediate successor of *a*, then *c(a)* Š *c(b)*
- Deciding that enough solutions have been generated
  - maximum number of solutions
  - minimum difference between cost of best feasible solution and the cost of the best assignment on the *Agenda*
  - minimum difference between costs of the last two assignments
- *Agenda* management as a priority queue

Representing assignments

- Each assignment is represented by the set of variable values that *differ* from the least cost assignment

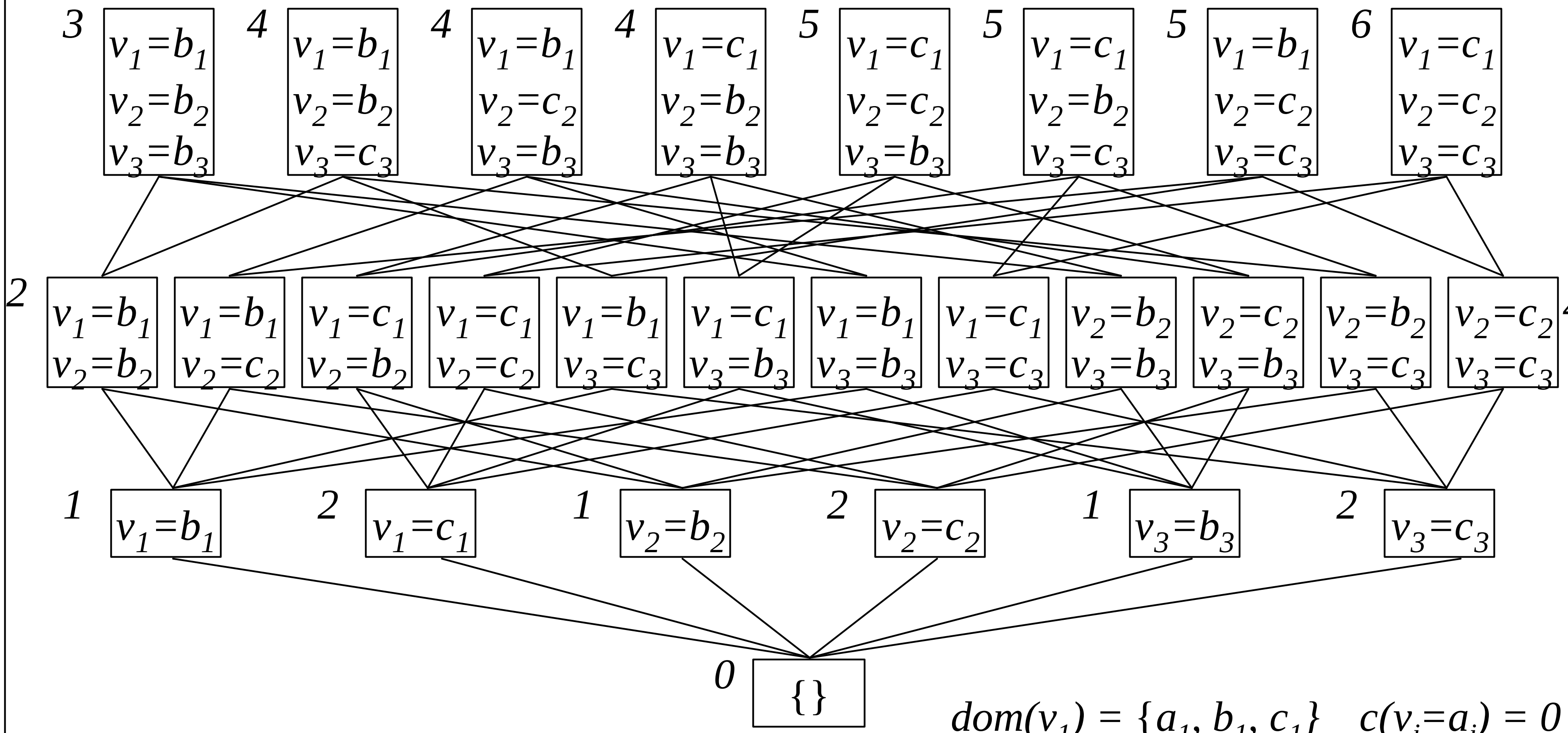
$$
\begin{array}{ll}
dom(v_1) = \{a_1, b_1, c_1\} & c(v_i=a_i) = 0 \\
dom(v_2) = \{a_2, b_2, c_2\} & c(v_i=b_i) = 1 \\
dom(v_3) = \{a_3, b_3, c_3\} & c(v_i=c_i) = 2
\end{array}
$$

- Least cost assignment $\{v_1=a_1, v_2=a_2, v_3=a_3\}$
- Assignment $\{v_1=a_1, v_2=a_2, v_3=b_3\}$ represented as just $\{v_3=b_3\}$

Basic successor function

- Assignment A_2 is an *immediate* successor of assignment A_1 if
  - the representation of A_1 is a *subset* of the representation of A_2; and
  - the representations of A_1 and A_2 differ by exactly one variable value
  - *e.g.*, $\{v_3=b_3\}$ is an immediate successor of {}
  - *e.g.*, $\{v_3=b_3 , v_2=b_2\}$ is an *eventual* successor, but *not* an immediate successor, of {}
- Definition of immediate successors is
  - *complete*: all assignments are eventual successors of the least cost assignment
  - *monotonic*: if A_2 is an immediate successor of A_1, then $c(A_1)$ Š $c(A_2)$

Successor lattice



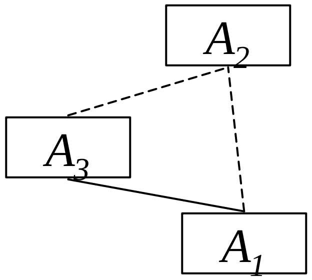
dom(v_1) = {a_1, b_1, c_1} $c(v_i=a_i) = 0$
dom(v_2) = {a_2, b_2, c_2} $c(v_i=b_i) = 1$
dom(v_3) = {a_3, b_3, c_3} $c(v_i=c_i) = 2$

Conflicts

- A *conflict* is a *partial* assignment that is guaranteed to be infeasible
  - any assignment that *contains* (or is *subsumed* by) a conflict is infeasible
  - [Davis 84; Genesereth 84; de Kleer & Williams 87]
  - *e.g.*, if the partial assignment $\{v_3=a_3, v_2=a_2\}$ is a conflict, then the assignment $\{v_3=a_3, v_2=a_2, v_1=b_1\}$ is infeasible
- *Requirement*: whenever *f* determines that an assignment is infeasible, it returns a conflict
  - if assignment *A* is infeasible, then *A* itself is trivially a conflict
  - ideally, *f* should return a *minimal* infeasible subset of *A* as a conflict
  - conflicts can be generated using dependency tracking in a truth maintenance system

Focusing with conflicts

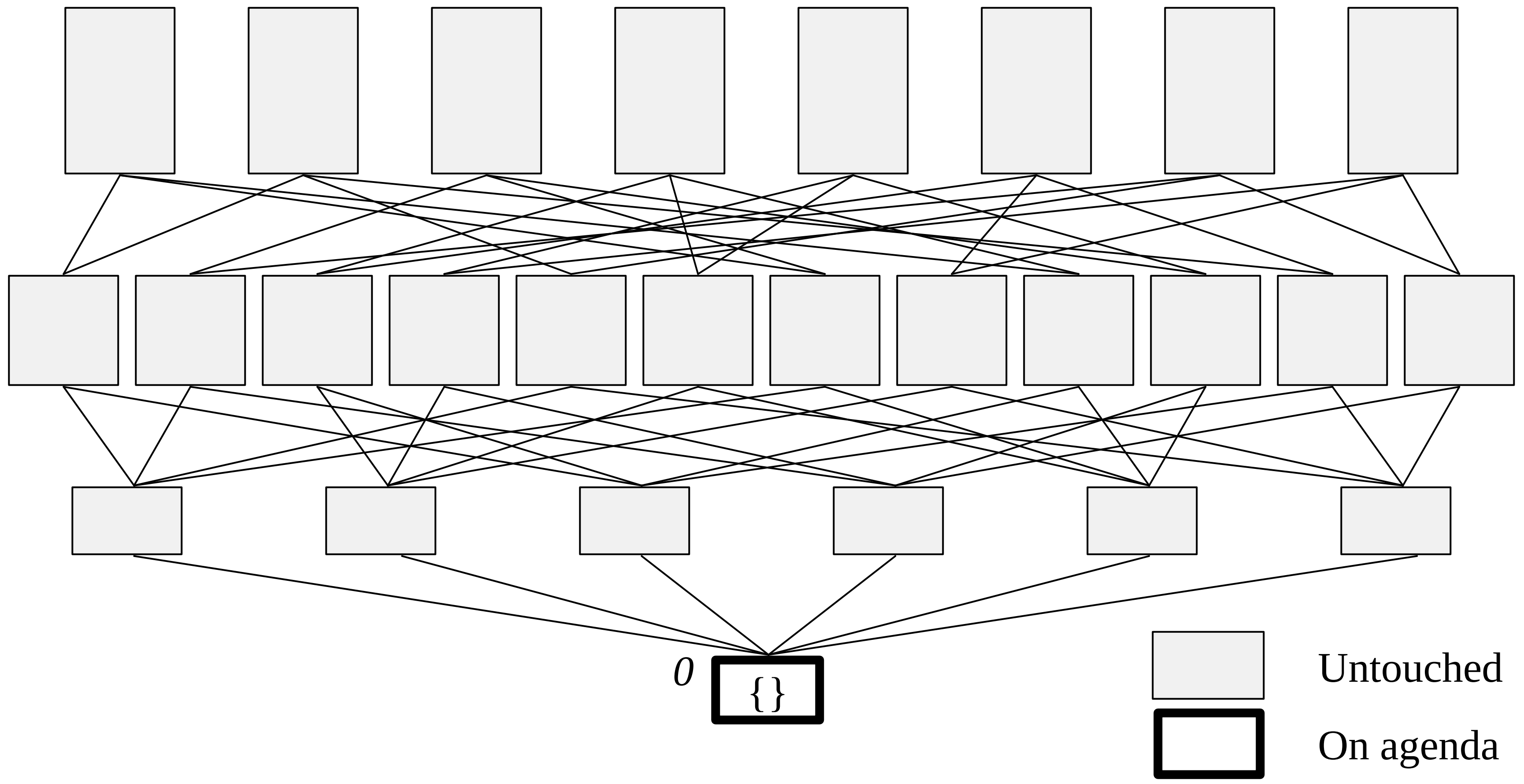
- *Lemma*: Let A_2 be an (eventual) successor of A_1 such that A_1 is subsumed by a conflict N, but A_2 is not. Then there exists an immediate successor A_3 of A_1 that is not subsumed by N such that A_2 is an (eventual) successor of A_3.



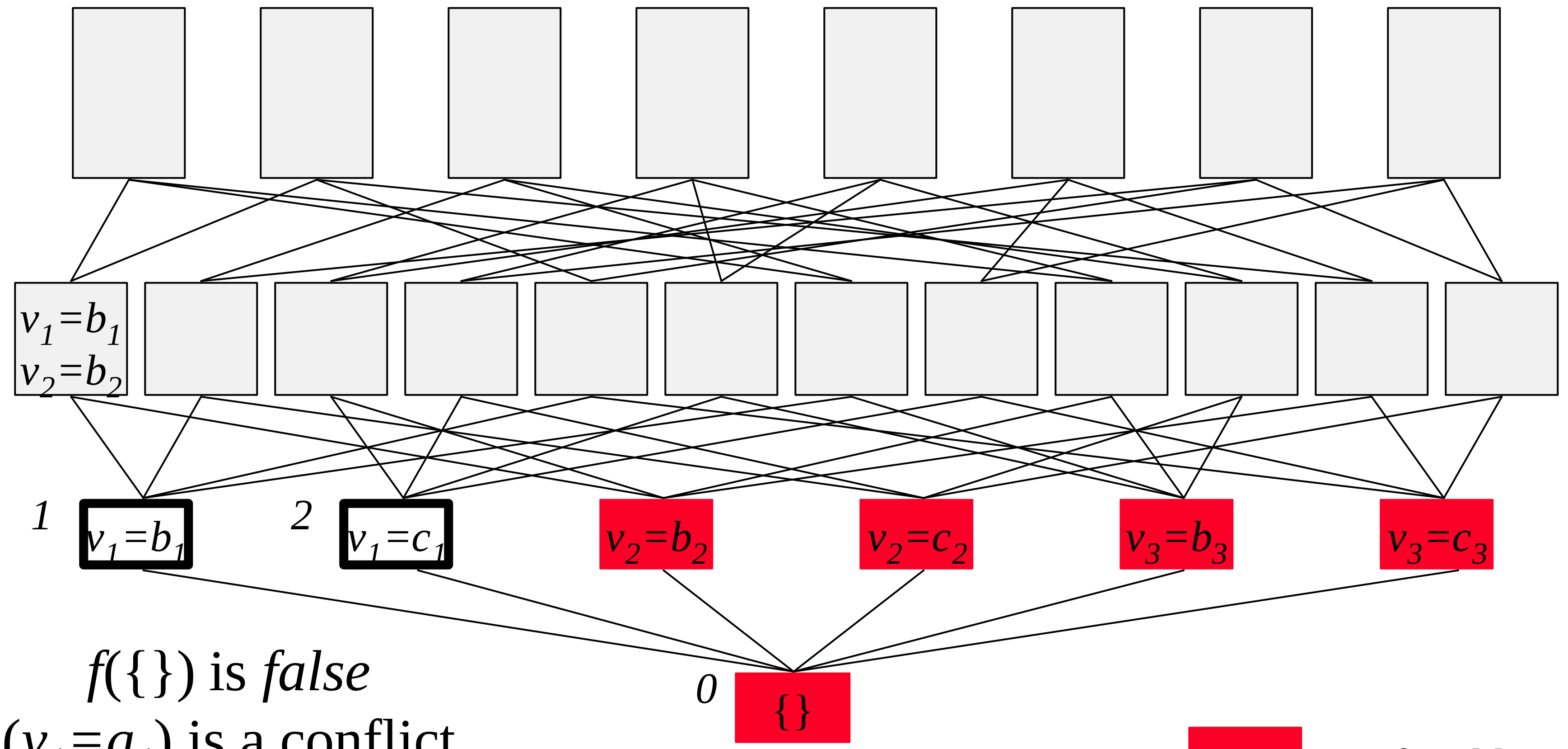
$\Rightarrow$ If an assignment A_1 is infeasible and is subsumed by a conflict N, then we need only generate those immediate successors of A_1 that are *not* subsumed by N

- the lemma ensures that completeness is preserved
- the smaller the conflict, the fewer the immediate successors

Initializing the agenda



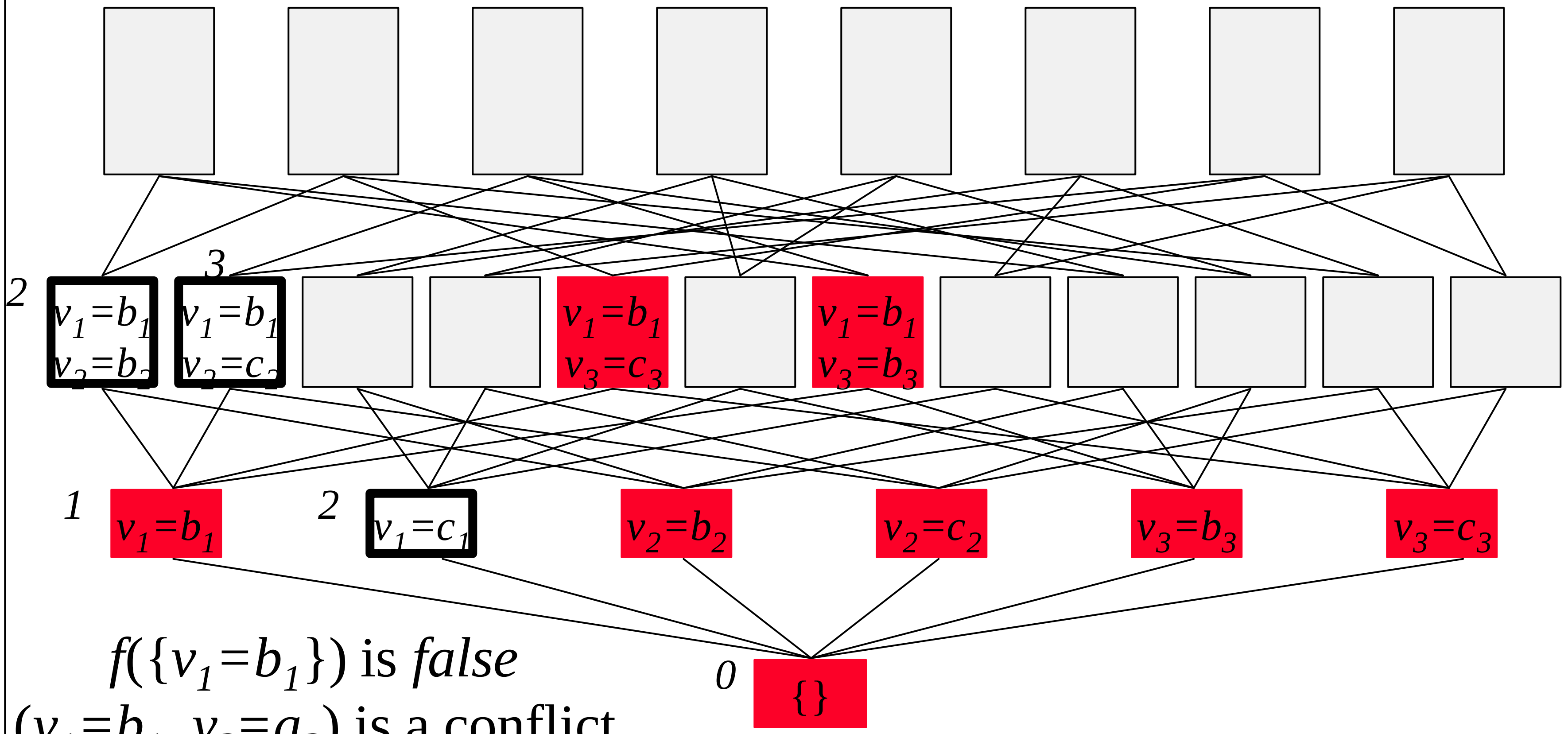
Assignment {} is infeasible



f({}) is *false*
($v_1=a_1$) is a conflict

🟥 Infeasible

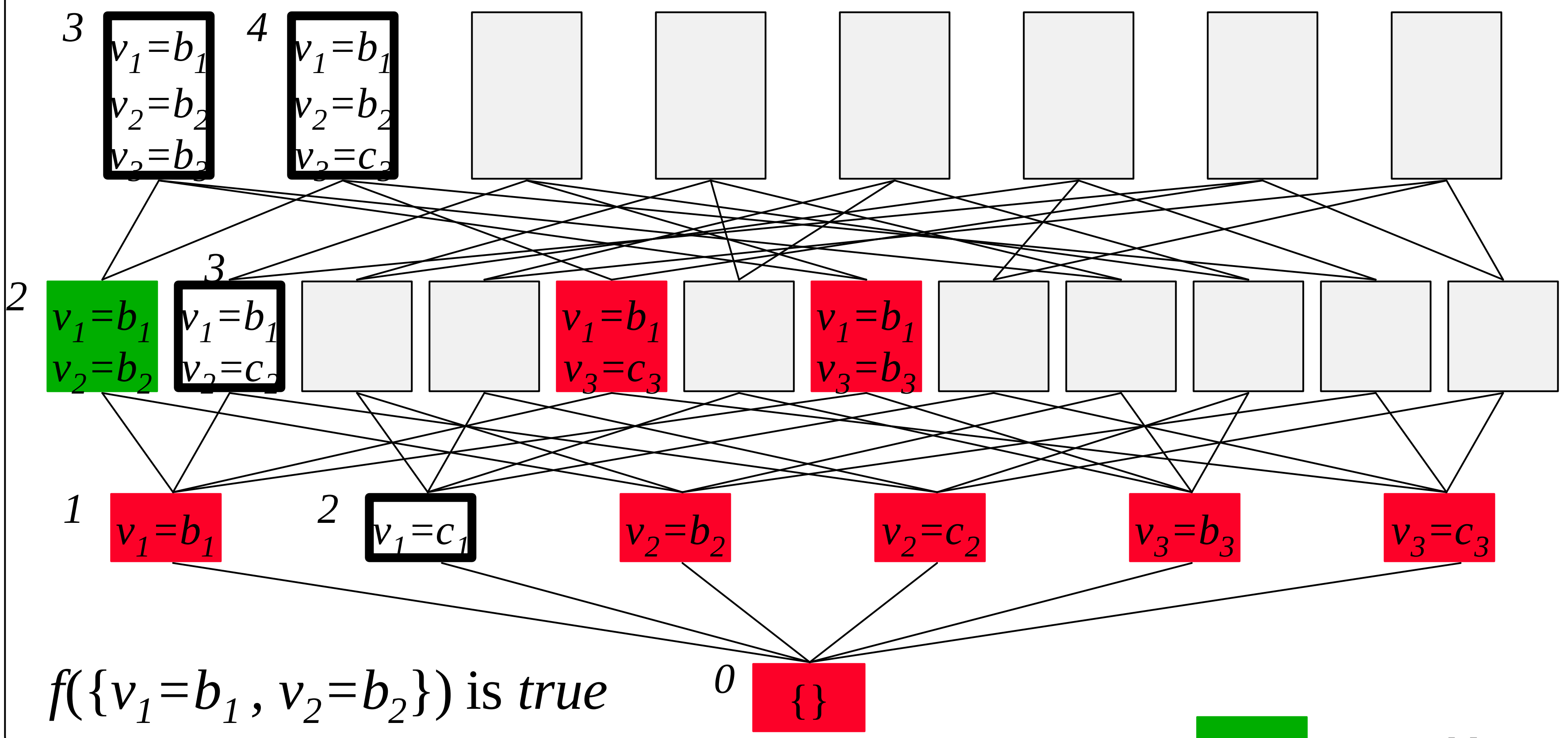
Assignment $\{v_1=b_1\}$ is infeasible



$f(\{v_1=b_1\})$ is *false*

$(v_1=b_1, v_2=a_2)$ is a conflict

Least cost feasible assignment found



$f(\{v_1=b_1 , v_2=b_2\})$ is *true*

🟩 Feasible

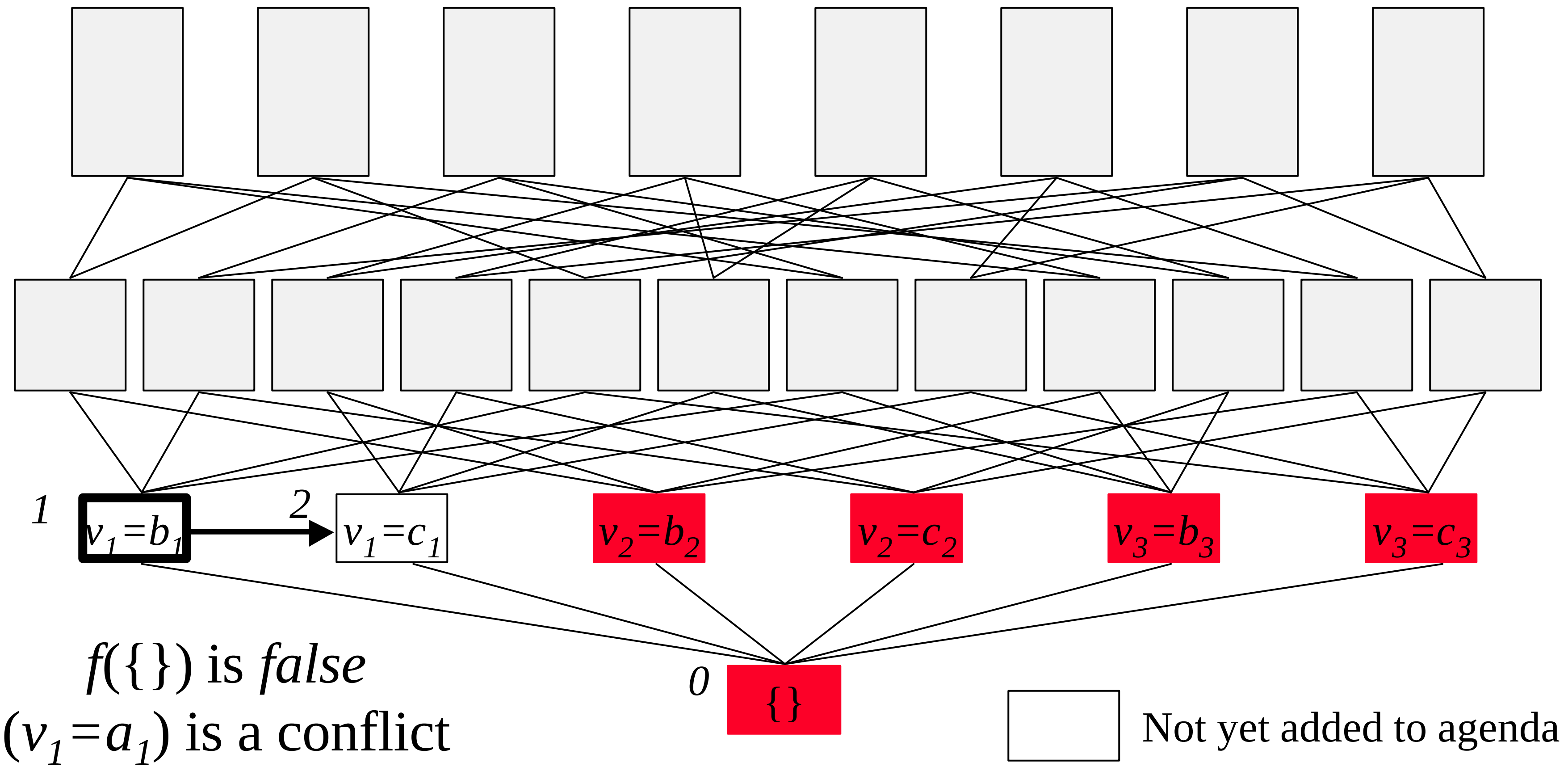
Decreasing agenda size

- Agenda size can be problematic in a best first search
  - for a branching factor *b*, agenda grows to size *O(bk)* after *k* checks
  - inserting *b* elements into the agenda after *k* checks is *O(b logb+b logk)*
- Immediate successors of an assignment are totally ordered
  - non-least cost successors only checked *after* least cost successor

$\Rightarrow$ Insert only least cost successor onto agenda
Sort remaining successors
Each assignment has exactly two successors

  - least cost immediate successor
  - next more expensive sibling
- Size of the agenda is *bounded by* the number of checks
  - inserting *b* successors after *k* checks is *O(b logb + 2logk)*

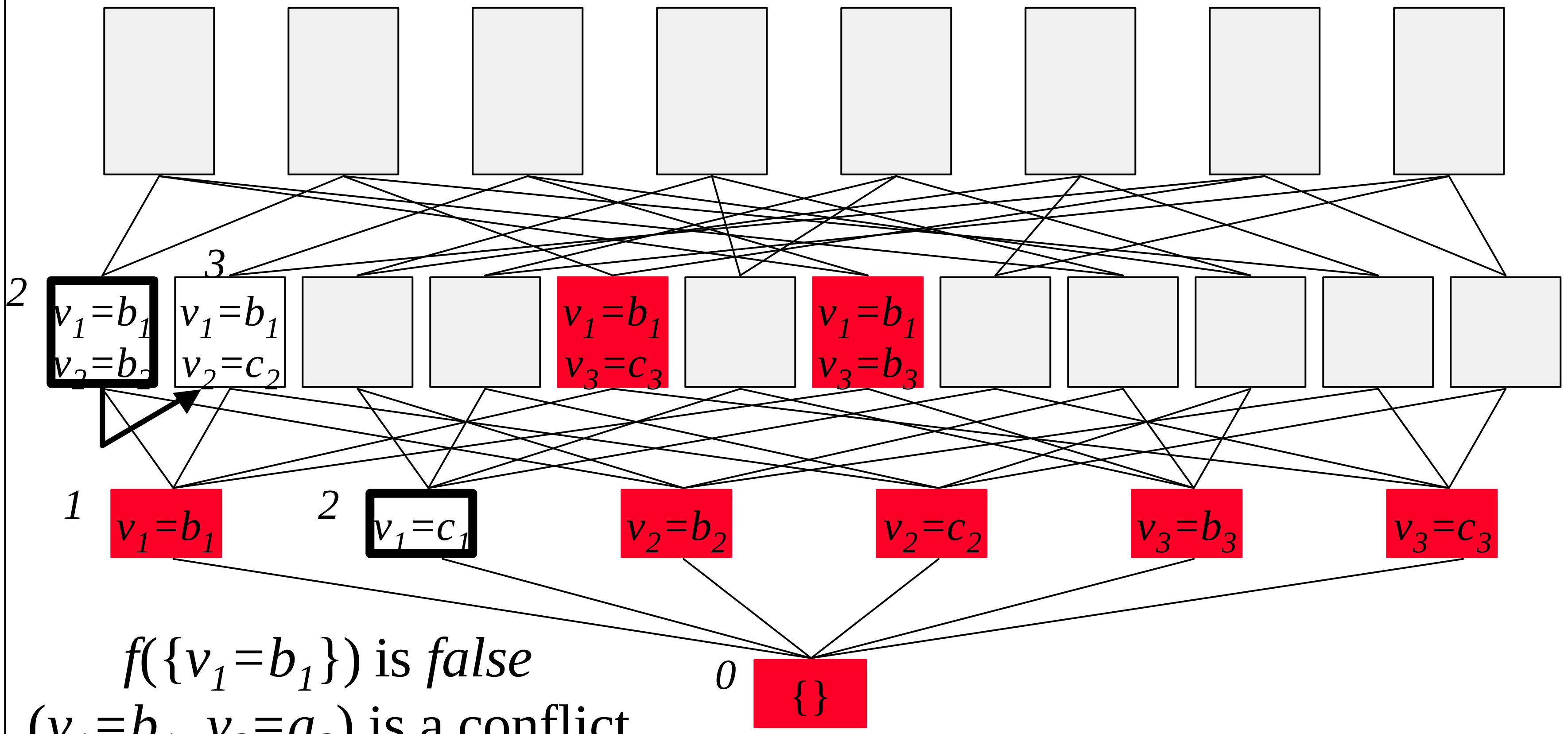
Only $\{v_1=b_1\}$ added to agenda



f({}) is *false*

($v_1=a_1$) is a conflict

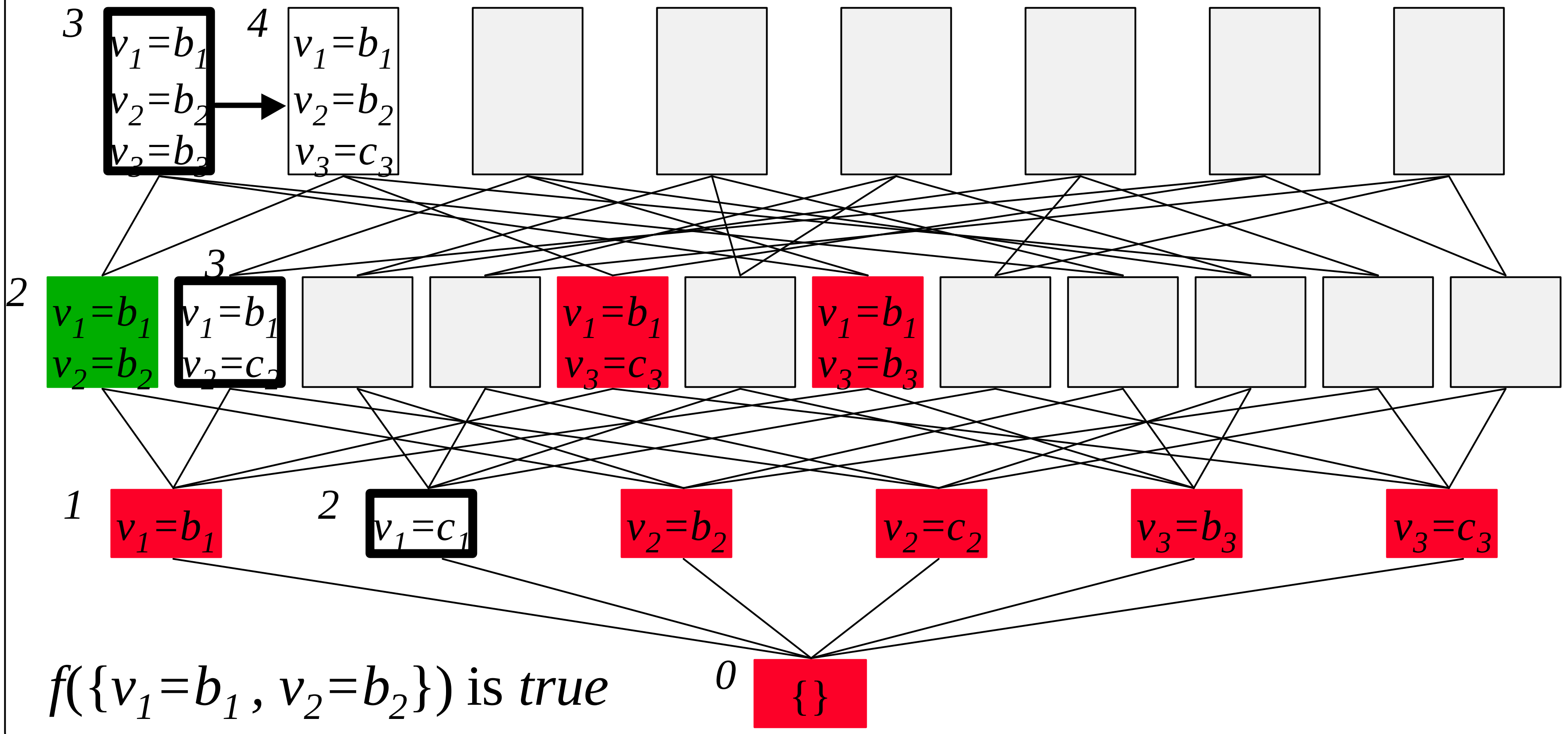
Immediate successor and sibling of $\{v_1=b_1\}$ added to agenda



$f(\{v_1=b_1\})$ is *false*

$(v_1=b_1, v_2=a_2)$ is a conflict

Least cost feasible assignment found



$f(\{v_1=b_1, v_2=b_2\})$ is *true*